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# AN EXTENSION OF FEUERBACH'S THEOREM

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Feuerbach's theorem, that the four circles which touch three lines also touch a circle, may be stated thus: given four orthocentric points, forming four triangles, the 16 in-circles of these triangles touch the circle  $F$  on the diagonal points.

Now each in-circle and the omitted one of the four points is a degenerate curve of class three on the absolute points  $IJ$ . There is further a rational curve of class three, on the six joins of the four points and touching the line infinity at  $IJ$ , which touches  $F$  three times. Thus the theorem is suggested: *All circular line-cubics on the joins of four orthocentric points touch the Feuerbach circle.*

A proof is as follows. It is convenient to state the algebra dually. That is, we have 4 lines  $1, \neq 1, \neq 1$  and a pair of lines,  $\zeta$  and  $1/\zeta$ , apolar to all conics on the 4 lines. Two point-cubics on the six joins of the 4 lines meet again at 3 points  $xyz$ , which are points of contact of a tritangent conic of either cubic. When  $x$  and  $y$  are given,  $z$  is rationally known; and when  $x$  is given and  $z$  moves on a line  $\zeta$ , we know from the theory of the Geiser transformation that  $y$  moves on a rational quartic  $\rho_x^4$  which has a triple point at  $x$ . There is then a connex of the form

$$\zeta x^4 y^4$$

where  $z$  is on  $\zeta$ . And if  $\xi$  be the join of  $x$  and  $y$  this connex is of the form

$$\zeta \xi^3 x y. \tag{1}$$

If  $\zeta$  be  $1, 0, 0$ , the quartic in  $y$  is the two lines

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ 0 & 1 & \neq 1 \end{vmatrix}$$

and the conic on  $x$  and the 4 other points, that is

$$(\xi_1^2 - \xi_2^2) \begin{vmatrix} x_0^2 - x_1^2 - x_2^2 & x_1 x_2 \\ y_0^2 - y_1^2 - y_2^2 & y_1 y_2 \end{vmatrix}.$$

Hence the connex is explicitly

$$\begin{aligned} \sum \zeta_0 (\xi_1^2 - \xi_2^2) \{ (x_0 y_1 + x_1 y_0) \xi_1 - (x_0 y_2 + x_2 y_0) \xi_2 \\ + 2 (x_1 y_1 - x_2 y_2) \xi_0 \} = 0. \end{aligned} \tag{2}$$

We now find where this curve in  $y$  meets the line  $1/\zeta$ . That is, we eliminate  $y$  from

$$\Sigma = 0, (\xi') = 0, (y/\zeta) = 0.$$

We have then

$$\Sigma \zeta_0 (\xi_1^2 - \xi_2^2) \begin{vmatrix} x_1 \xi_1 - x_2 \xi_2, & x_0 \xi_1 + 2x_1 \xi_0, & -x_0 \xi_2 - 2x_2 \xi_0 \\ \xi_0 & \xi_1 & \xi_2 \\ 1/\zeta_0 & 1/\zeta_1 & 1/\zeta_2 \end{vmatrix} = 0,$$

or since  $(x\xi) = 0$

$$\Sigma (\zeta_1/\zeta_2 - \zeta_2/\zeta_1) x_0 (\xi_0^2 - \xi_1^2) (\xi_0^2 - \xi_2^2) = 0$$

or if  $a$  be the join of  $\zeta$  and  $1/\zeta$ ,

$$\Sigma a_0 x_0 (\xi_0^2 - \xi_1^2) (\xi_0^2 - \xi_2^2) = 0. \quad (3)$$

The values of  $\xi$  common to this equation and  $(x\xi) = 0$  give the intersections  $y$  of  $\rho_x^4$  and  $1/\zeta$ . Thus when  $(x\xi)$  is a line of (3) then as  $z$  moves on  $\zeta$  the curve  $\rho_x^4$  touches  $1/\zeta$  at the point  $y$ , and  $x$  is on the envelope sought.

Now the quartic (3) is two conics on the lines  $1, \pm 1, \pm 1$ . And when

$$\Sigma \sqrt{a_i x_i} = 0, \quad (4)$$

the two conics become one conic  $R$  whose equation is

$$\Sigma \sqrt{a_i x_i} \xi_i^2. \quad (5)$$

The conic (4) occurs then twice in the envelope, the other factors being

$$\begin{aligned} [x_0^4 + 2x_1^2 x_2^2]^2, \\ x_0 x_1 x_2, \end{aligned}$$

and the cubic of the system with a double point at  $a$ , namely

$$\Sigma x_0 \eta_1 \eta_2$$

where  $\eta$  is the join of  $x$  and  $a$ .

The conic (4) is the Feuerbach conic  $F$ , for it is on the diagonal lines of the four lines, and having the line equation

$$\Sigma a/\xi = 0,$$

it is on  $\zeta$  and  $1/\zeta$ .

The construction of the cubics which touch both  $\zeta$  and  $1/\zeta$  is then as follows. Take a point  $x$  on  $F$ , and draw from  $x$  the two tangents to the conic  $R$ . The diagonals of this line-pair and the pair  $\zeta$  and  $1/\zeta$  give the points of contact of the two cubics. If the diagonals meet at  $b$ ,

then  $x$  and  $b$  are apolar to  $\zeta$  and  $1/\zeta$ ; and the line  $bx$ , being the polar of  $a$  as to  $R$ , has the equation

$$\Sigma ay/\sqrt{ax} = 0,$$

and is the tangent of  $F$  at  $x$ .

Dually then,  $\zeta$  and  $1/\zeta$  being the absolute points, the conic  $F$  the Feuerbach circle, and the conic  $R$  a rectangular hyperbola on the four given orthocentric points, and having its centre  $c$  on  $F$ , if the common diameter of  $F$  and  $R$  meet  $R$  at points  $dd'$ , then these points are double foci of circular curves of class 3 on the 6 lines; the circles with centres  $d$  and  $d'$  and touching  $F$  at  $c$  are the tritangent conics; and the two cubics touch  $F$  at  $c$ .

## DEFORMATIONS OF TRANSFORMATIONS OF RIBAUCCOUR

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When a system of spheres involves two parameters, their envelope consists in general of two sheets, say  $\Sigma$  and  $\Sigma_1$ , and the centers of the spheres lie upon a surface  $S$ . A correspondence between  $\Sigma$  and  $\Sigma_1$  is established by making correspond the points of contact of the same sphere. In general the lines of curvature on  $\Sigma$  and  $\Sigma_1$  do not correspond. When they do, we say that  $\Sigma_1$  is in the relation of a transformation of Ribaucour with  $\Sigma$ , and vice-versa. For the sake of brevity we call it a transformation  $R$ .

It is a known property of envelopes of spheres that if the surface of centers  $S$  be deformed and the spheres be carried along in the deformation, the points of contact of the spheres with their envelope in the new position are the same as before deformation.<sup>1</sup> Ordinarily when  $S$  for a transformation  $R$  is deformed, the new surfaces  $\Sigma'$  and  $\Sigma'_1$  are not in the relation of a transformation  $R$ . Bianchi<sup>2</sup> has shown that when  $S$  is applicable to a surface of revolution, it is possible to choose spheres so that for every deformation of  $S$  the two sheets of the envelopes of the spheres shall be in the relation of a transformation  $R$ , and this is the only case in which  $S$  can be deformed continuously with transformations  $R$  preserved. The only other possibility is that in which it is possible to deform the surface of centers of a transformation  $R$  in one way so that the sheets of the new envelope shall be in the relation of a transformation  $R$ . It is the purpose of this paper to determine this class of transformations  $R$ .